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FREE ACTIONS OF SURFACE GROUPS ON R-TREES

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Dedicated to the memory of Roger Lyndon

IN [9] Lyndon introduced a class of real-valued functions on groups, now called Lyndon length functions, and showed that a group is free if and only if it admits an integer-valued Lyndon length function. He raised the question of which groups admit **R**-valued Lyndon length functions. Using the construction of Chiswell [3] this question can be re-interpreted as asking which groups act freely (by isometries) on **R**-trees. (For the definition of an **R**-tree see [12].)

The examples of such groups pointed out by Lyndon are arbitrary free products of subgroups of **R**. The first examples not of this type were given by Alperin–Moss in [2]. Their examples are not finitely generated. In this paper we give the first finitely generated examples which are not free products of free abelian groups.

THEOREM 1. *Let G be a surface group, i.e., the fundamental group of a closed surface. Then there exist an* **R**-*tree T and a free action* $G \times T \to T$ *unless G is isomorphic to the fundamental group of a closed, non-orientable surface of Euler characteristic* ≥ -1.

The exceptions in Theorem 1 are genuine:

THEOREM 2. *If G is isomorphic to the fundamental group of a closed non-orientable surface of Euler characteristic* ≥ -1, *then G does not act freely on any* **R**-*tree.*

We refer to the three closed non-orientable surfaces of Euler characteristic ≥ -1 as *exceptional surfaces*. Their fundamental groups are called *exceptional surface groups*.

The class of groups that act freely on **R**-trees is closed under free products. Thus, it follows from Theorem 1 that any free product of non-exceptional surface groups and finitely generated free abelian groups acts freely on an **R**-tree. This raises the following questions:

If Γ *is a finitely generated group which acts freely on an* **R**-*tree T, then is* Γ *a free product of non-exceptional surface groups and finitely generated free abelian groups?*

According to Proposition IV.2.2 of [14], the above question has a affirmative answer if Γ is assumed to be the fundamental group of a 3-manifold. In [10] it is proved that the answer is affirmative if Γ is assumed to be of the form $F_0 *_{\mathbf{Z}} F_1$ where F_0 and F_1 are free groups. In [6] it is shown that the answer is affirmative in the case when T is the completion of a Λ-tree where Λ is a subgroup of **R** with $\dim_{\mathbf{Q}}(\Lambda \otimes \mathbf{Q}) \leq 2$.

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The proof of Theorem 1 is immediate in the case when the surface is not hyperbolic. In the hyperbolic case it is a consequence of Thurston's theory of measured laminations. The point is that on any non-exceptional closed hyperbolic surface there is a transversely measured, geodesic lamination all of whose complementary components are simply connected. We show that dual to a transversely measured, geodesic lamination on a surface there is an action of the fundamental group of the surface on an **R**-tree. The stabilizers of the points of the tree are the stabilizers of the complementary regions and of the leaves of the induced lamination in the universal covering. Thus, the condition that the complementary regions be simply connected (and consequently, that all the leaves be simply connected) implies that the dual action is free.

The proof of Theorem 2 is also immediate in the case when the surface in question is $\mathbf{RP}^2$ or a Klein bottle. The case of the closed surface of Euler characteristic -1 follows an analysis inspired by that of [10].

We first announced Theorem 1 in the orientable case in 1982. It was incorrectly asserted in [15] that Theorem 1 holds for all surfaces except $\mathbf{RP}^2$ and the Klein bottle. We are grateful to Walter Parry for pointing out to us the correct statement of Theorem 1 in the non-orientable case. He also gave an independent proof in this case.

1. THE R-TREE DUAL TO A MEASURED LAMINATION

This section is devoted to constructing **R**-trees dual to certain measured laminations in manifolds, and to describing the stabilizers of points under these actions in terms of the original laminations. At the end of the section we specialize to the case of a measured geodesic lamination on a hyperbolic surface.

By a measured lamination in a compact manifold we mean a codimension-1 lamination with a transverse measure of full support. For a more detailed definition and for the basic properties of measured laminations see [13].

In this section $(\mathscr{L}, \mu)$ is a measured lamination, with support X, in a compact manifold M. Let $(\tilde{\mathscr{L}}, \tilde{\mu})$ be the induced measured lamination in the universal covering $\tilde{M}$. Let C be the set of components of $\tilde{M} - \tilde{X}$. The following conditions will be assumed to hold:

(3a) X is nowhere dense in M;
(3b) each leaf of $\tilde{\mathscr{L}}$ is a closed subset of $\tilde{M}$; and
(3c) if p and q are points of $\tilde{M} - \tilde{X}$ then there is a path ω; $I \to \tilde{M}$, with $\omega(0) = p$ and $\omega(1) = q$, which is transverse to the leaves of $\tilde{\mathscr{L}}$ and which crosses each leaf at most once.

Let $c_0, c_1 \in C$. Suppose that ω and ω' are transverse paths from points $x_0 \in c_0$ to $x_1 \in c_1$ and from $x'_0 \in c_0$ to $x'_1 \in c_1$ respectively with ω crossing each leaf of $\tilde{\mathscr{L}}$ at most once. There is a map Ω: $I \times I \to \tilde{M}$ with $\Omega|I \times \{0\} = \omega$, $\Omega|I \times \{1\} = \omega'$, $\Omega(\{0\} \times I) \subset c_0$ and $\Omega(\{1\} \times I) \subset c_1$. Since $\tilde{\mathscr{L}}$ is nowhere dense, we can suppose that Ω is transverse to $(\tilde{\mathscr{L}}, \tilde{\mu})$. Then Ω pulls $\tilde{\mathscr{L}}$ back to a measured lamination $\Omega^*(\tilde{\mathscr{L}}, \tilde{\mu})$ on $I \times I$. By Condition (3b), all leaves of $\Omega^* \tilde{\mathscr{L}}$ are compact. Clearly then, each leaf of $\Omega^* \tilde{\mathscr{L}}$ is either a circle or an arc with endpoints in $I \times \{0, 1\}$. No leaf of $\Omega^* \tilde{\mathscr{L}}$ can have both endpoints in $I \times \{0\}$ since ω crosses each leaf of $\tilde{\mathscr{L}}$ at most once. It follows that all the leaves of $\Omega^* \tilde{\mathscr{L}}$ which meet $\partial(I \times I)$ are either arcs running from $I \times \{0\}$ to $I \times \{1\}$ or arcs with both endpoints in $I \times \{1\}$. From this one sees easily that

$$\int_I \omega^*(\tilde{\mu}) \leq \int_I (\omega')^*(\tilde{\mu}) \tag{4}$$

with equality if and only if ω' crosses each leaf of $\tilde{\mathscr{L}}$ at most once.

We define a metric on the set C as follows: Let c_0 and c_1 be points of C, i.e., complementary regions of $\tilde{\mathscr{L}}$ in $\tilde{M}$. Choose points $x_0 \in c_0 \subset \tilde{M}$ and $x_1 \in c_1 \subset \tilde{M}$. Let ω: $I \to \tilde{M}$ be a path transverse to the leaves of $\tilde{\mathscr{L}}$ with $\omega(0) = x_0$ and $\omega(1) = x_1$. Suppose furthermore that ω crosses each leaf of $\tilde{\mathscr{L}}$ at most once. We define $d(c_0, c_1)$ to be the geometric intersection number of ω with $(\tilde{\mathscr{L}}, \tilde{\mu})$, i.e., to be $\int_I \omega^*(\tilde{\mu})$. It follows from (4) that d is well-defined and satisfies the triangle inequality. It is clear from the definition that d is $\pi_1(M)$-invariant.

LEMMA 5. *There exist an* **R**-*tree* T *and an isometric embedding* ψ: $C \hookrightarrow T$ *such that*:

(i) $\psi(C)$ *spans* T (*i.e., the smallest subtree of* T *containing* $\psi(C)$ *is all of* T).

(ii) *Any point* $t \in T - \psi(C)$ *is an edgepoint in the sense that* t *separates* T *into two components.*

(iii) *The action of* $\pi_1(M)$ *on* C *extends uniquely to an action of* $\pi_1(M)$ *by isometries on* T. *Between any two such* **R**-*trees there is an equivariant isometry commuting with the embeddings of* C.

Proof. According to [1], to show the existence of an **R**-tree T and an isometric embedding ψ: $C \hookrightarrow T$ satisfying Condition (i), we need only show that the *overlap function* $p \wedge_r q$ defined by $p \wedge_r q = (d(p, r) + d(q, r) - d(p, q))/2$ satisfies the following property:

$$(p \wedge_r q) > (p \wedge_r s) \Rightarrow p \wedge_r s = q \wedge_r s. \tag{6}$$

In order to verify (6) we give a geometric interpretation of $p \wedge_r q$. Choose points $x_0 \in r$, $x_1 \in p$ and $x_2 \in q$. Let ω_{ij} be a path transverse to $\tilde{\mathscr{L}}$, crossing each leaf at most once, beginning at x_i and ending at x_j. Let σ be an abstract 2-simplex with vertices $\{x_0, x_1, x_2\}$. The ω_{ij} define a map $\partial\sigma \to \tilde{M}$. We extend this map to a map Ω: $\sigma \to \tilde{M}$ transverse to $\tilde{\mathscr{L}}$. The lamination $\Omega^* \tilde{\mathscr{L}}$ has the form suggested by Fig. 1.

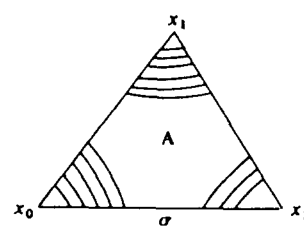

Fig. 1

The total measure of the leaves which are arcs in σ joining a point of $[x_0, x_1]$ to a point of $[x_0, x_2]$ is $p \wedge_r q$. Let c be the component of $\tilde{M} - \tilde{\mathscr{L}}$ containing $\Omega(A)$. The paths ω_{01} and ω_{02} enter c through the same boundary component, but since ω_{12} crosses each leaf of $\tilde{\mathscr{L}}$ at most once, they leave c through distinct boundary components.

Since each leaf of $\tilde{\mathscr{L}}$ separates $\tilde{M}$ into two components and since the ω_{0j} cross no leaf twice, we see that ω_{01} and ω_{02} cross the same set of complementary regions up to c and then afterwards cross no region in common.

Now suppose that $(p \wedge_r q) > (p \wedge_r s)$. Let ω_{01}, ω_{02} and ω_{03} be paths transverse to $\tilde{\mathscr{L}}$ from r to p, from r to q and from r to s respectively. Then as we move along ω_{01} the last region that ω_{01} and ω_{03} cross in common comes before the last region that ω_{01} and ω_{02} cross in common. Thus the last region that ω_{03} and ω_{02} cross in common is equal to the last region that ω_{01} and ω_{03} cross in common. This proves that $p \wedge_r s = q \wedge_r s$, and thus, establishes (6) above.

It follows from [1] that if $\psi\colon C \to T$ and $\psi'\colon C \to T'$ are two isometric embeddings satisfying (i), then there is a unique isometry $j\colon T \to T'$ such that $j \circ \psi = \psi'$. This implies (iii).

Notice that it also follows from our geometric description of the overlap function that there is a point $c \in C$ such that $p \wedge_r q = d(p, c)$ and $d(p, c) + d(c, q) = d(p, q)$. According to [1] this means that the intersection of any two segments in T with endpoints in $\psi(C)$ and the same initial point is itself a segment with endpoints in $\psi(C)$. This establishes (ii). □

If $(\mathscr{L}, \mu)$ satisfies Conditions (3), then the **R**-tree given by Lemma 5 will be called the *dual tree*. Implicitly, the dual tree is equipped with the action given in Part (iii) of Lemma 5. Let us now consider the stabilizers in $\pi_1(M)$ of the various points of the dual tree.

PROPOSITION 7. *Let $(\mathscr{L}, \mu)$ satisfy Conditions (3). Let T denote the dual tree. For any $c \in C$ the stabilizer in $\pi_1(M)$ of $\psi(c) \in T$ is equal to the stabilizer of $c \subset \tilde{M}$. For any $t \in T - \psi(C)$ there is a leaf of $\tilde{\mathscr{L}}$ whose stabilizer is equal to that of t.*

Proof. The first part is clear. Now suppose that $t \in T - \psi(C)$. Since by Lemma 5, $\psi(C)$ spans T and t separates T into two components, there are points $a, b \in \psi(C)$ on opposite sides of t. Set $c_a = \psi^{-1}(a)$ and $c_b = \psi^{-1}(b)$. Let ω be a path from c_a to c_b crossing each leaf of $\tilde{\mathscr{L}}$ at most once. The least upper bound of the set of points in $[0, 1]$ mapped by ω to complementary regions corresponding to points of T on the a-side of t is equal to the greatest lower bound of the set of points in $[0, 1]$ mapped to complementary regions corresponding to points of T on the b-side of t. This common point is mapped to a leaf of $\tilde{\mathscr{L}}$ which separates the two sets of complementary regions. The stabilizer of this leaf is the stabilizer of $t \in T$. □

Now let Σ be a (complete) hyperbolic surface of finite area. A *geodesic lamination* on Σ is a lamination all of whose leaves are geodesics. A *measured geodesic lamination* on Σ is a geodesic lamination with a transverse measure of full support.

LEMMA 8. *Any measured geodesic lamination on a complete hyperbolic surface Σ of finite area satisfies Conditions (3) and therefore has a dual tree.*

Proof. It is shown in [16, p. 8.27] that the support of a geodesic lamination on Σ is of zero 2-dimensional Hausdorff measure and hence is nowhere dense. Clearly, the lift of a geodesic on a hyperbolic surface to the universal covering is a closed subset. Finally, if a and b are points in the complement of a geodesic lamination, then the geodesic arc between them meets every geodesic (except the one that it lies on) in at most one point. Hence it crosses each leaf of the given geodesic lamination at most once. □

2. THE SPACE OF PROJECTIVE GEODESIC LAMINATIONS

Theorem 1 will be proved in Section 3 by applying Lemma 8 to a measured geodesic lamination all of whose leaves and complementary regions are simply connected. The construction of such a measured geodesic lamination depends on some of the basic properties of Thurston's space of projective geodesic laminations (see Section 8.10 of [16] and [17] pp. 373–375) which we shall review in this section.

Let Σ be a hyperbolic surface of finite area. We identify the universal cover of Σ with the Poincaré disk **H**. A geodesic in **H** is determined by its endpoints in the unit circle. Thus we may identify the set $\mathscr{G}$ of all geodesics in **H** with

$$(S^1 \times S^1 - \Delta)/\tau$$

where Δ is the diagonal and τ is the involution which interchanges the factors. This gives $\mathscr{G}$ a natural topology in which it is homeomorphic to the open Möbius band. The action of $\pi_1(\Sigma)$ on $\mathbf{H}$ induces an action on $\mathscr{G}$. Since Σ is of finite area, there is a compact subset F of $\mathscr{G}$ such that the translates of int F under $\pi_1(\Sigma)$ cover $\mathscr{G}$.

Geodesic laminations on Σ are in one-to-one correspondence with closed subsets $L \subset \mathscr{G}$ satisfying the following two conditions.

(9a) L is invariant under $\pi_1(\Sigma)$.
(9b) If l_1 and l_2 are distinct geodesics in L then $l_1 \cap l_2 = \varnothing$.

A measured geodesic lamination on Σ corresponds to a locally finite Borel measure on $\mathscr{G}$ which is invariant under $\pi_1(\Sigma)$ and has as support a set L satisfying (9b). Thus the set $\mathscr{M}L(\Sigma)$ of all measured geodesic laminations on Σ can be identified with a subset of the space $B(\mathscr{G})$ of all locally finite Borel measures on $\mathscr{G}$. The weak-$*$ topology on $B(\mathscr{G})$ induces a natural topology on $\mathscr{M}L(\Sigma)$.

LEMMA 10. *$\mathscr{M}L(\Sigma)$ is a closed subset of $B(\mathscr{G})$.*

Proof. Clearly, invariance under $\pi_1(\Sigma)$ is a closed condition in $B(\mathscr{G})$. We show that the measures whose supports satisfy Condition (9b) form a closed subset of $B(\mathscr{G})$. Let μ be a measure whose support violates (9b). Then there are geodesics l_1, l_2 in the support of μ which cross. It follows from the definition of the topology on $\mathscr{G}$ that there are open neighborhoods U_i of the l_i such that for any $l'_1 \in U_1$ and $l'_2 \in U_2$ we have $l'_1 \cap l'_2 \neq \varnothing$. The measures that are positive on both U_1 and U_2 form an open neighborhood of μ in the weak-$*$ topology; all measures in this neighborhood violate (9b). □

Let $P(\mathscr{G})$ be the quotient of $B(\mathscr{G}) - \{0\}$ under the action of $\mathbf{R}^+$ by homotheties. With the quotient topology, $P(\mathscr{G})$ is a Hausdorff space. The set $\mathscr{M}L(\Sigma) - \{0\} \subset B(\mathscr{G}) - \{0\}$ is invariant under the $\mathbf{R}^+$-action. We denote its image in $P(\mathscr{G})$ by $\mathscr{P}L(\Sigma)$. An element of $\mathscr{P}L(\Sigma)$ is called a *projective geodesic lamination.* Note that a projective geodesic lamination has a well-defined underlying (geodesic) lamination. On the other hand, note that a simple closed geodesic γ is the underlying lamination of a unique projective geodesic lamination, which we shall denote by $P(\gamma)$. The support of $P(\gamma)$ is the set $L(\gamma)$ of all geodesics in $\mathbf{H}$ that cover γ.

PROPOSITION 11. *$\mathscr{P}L(\Sigma)$ is a compact Hausdorff space.*

N.B. Thurston [16, p. 8.60] and [17, Theorem 5.2] has shown that $\mathscr{P}L(\Sigma)$ is homeomorphic to the sphere of dimension $-3\chi(\Sigma) - 1$.

Proof. Since $P(\mathscr{G})$ is Hausdorff, so is $\mathscr{P}L(\Sigma)$.

Let $F \subset \mathscr{G}$ be a compact set such that the translates of int F under $\pi_1(\Sigma)$ cover $\mathscr{G}$. Let E denote the set of all $\pi_1(\Sigma)$-invariant measures $\mu \in B(\mathscr{G})$ such that $\mu(F) = 1$. Clearly, the natural continuous map

$$\rho \colon \mathscr{M}L(\Sigma) \cap E \to \mathscr{P}L(\Sigma)$$

is bijective. Hence, we need only show that $\mathscr{M}L(\Sigma) \cap E$ is compact. By Lemma 10 it is enough to show that E is compact.

Let E_0 denote the subset of $B(F)$ consisting of all measures ν of total mass 1 satisfying the following 'partial invariance' condition:

(12) For every $\gamma \in \pi_1(\Sigma)$ and every measurable set $A \subset F \cap \gamma^{-1}F$ we have $\nu(A) = \nu(\gamma A)$.

Every $v \in E_0$ extends uniquely to an $\pi_1(\Sigma)$-invariant locally finite measure on $\mathcal{G}$. This defines a natural continuous bijection $E_0 \to E$. But E_0 is clearly a closed subset of $B(F)$; since E_0 consists of measures of total mass 1, and since the unit ball in $B(F)$ is compact in the weak-$*$ topology, it follows that E_0 is compact. Hence E is compact. □

Remark 13. (a) It follows from the above argument that the map ρ is a homeomorphism. (b) For any simple closed geodesic γ in Σ, we may represent $P(\gamma)$ by

$$\sum_{l \in L(\gamma)} \delta_l$$

where δ_l is the δ-measure of mass 1 concentrated at l. Let us set $Q(\gamma) = \rho^{-1}(P(\gamma))|F$. Then

$$Q(\gamma) = \frac{1}{N(\gamma)} \sum_{l \in F(\gamma)} \delta_l$$

where $F(\gamma) = L(\gamma) \cap F$ is the set of geodesics in $\mathbf{H}$ covering γ and contained in F, and where $N(\gamma) = \# F(\gamma)$.

In addition to the above qualitative results about $\mathcal{PL}(\Sigma)$, we shall need a well-known fact. For completeness, we have included a proof of this result in an Appendix.

PROPOSITION 14. *Let β and γ be orientation-preserving simple closed geodesics in Σ meeting in one point. Let β_n be the simple closed geodesic freely homotopic to $\gamma^n * \beta$. Then*

$$\lim_{n \to \infty} P(\beta_n) = P(\gamma)$$

in $\mathcal{PL}(\Sigma)$.

3. PROOF OF THEOREM 1

Let Σ be a closed non-exceptional surface. Theorem 1 is obvious if Σ is not hyperbolic since $\pi_1(\Sigma)$ then is isomorphic to the trivial group or to $\mathbf{Z} \times \mathbf{Z}$. For the rest of this section Σ will be assumed to be a closed non-exceptional hyperbolic surface.

The method of proof is to produce a non-empty closed subset of $\mathcal{PL}(\Sigma)$ which contains a dense G_δ consisting of projective classes of measured laminations giving free actions of $\pi_1(\Sigma)$ on $\mathbf{R}$-trees. It is possible to show that this closed subset is the set of all projective geodesic laminations whose underlying geodesic laminations have no leaves which are orientation-reversing simple closed curves. In particular, in the case when Σ is orientable this closed subset is all of $\mathcal{PL}(\Sigma)$.

If Σ is orientable we let Z denote the set of simple closed geodesics which are non-separating. If Σ is non-orientable we let Z denote the set of simple closed geodesics which are non-separating, orientation-preserving, and have non-orientable complement. Thus in either case Z consists of all orientation-preserving simple closed geodesics which represent neither 0 nor w_1^* in $H_1(\Sigma; \mathbf{Z}/2\mathbf{Z})$, where w_1^* is the Poincaré dual of the first Stiefel–Whitney class. We consider the closure $\bar{Z}$ of Z in $\mathcal{PL}(\Sigma)$. Since $\mathcal{PL}(\Sigma)$ is a compact Hausdorff space, so is $\bar{Z}$. Clearly, $\bar{Z}$ is non-empty.

For any simple closed geodesic α in Σ we denote by U_α the subset of $\mathcal{PL}(\Sigma)$ consisting of all projective geodesic laminations which have at least one leaf crossing α. (In particular, a projective geodesic lamination which has α as a leaf does not belong to U_α.)

PROPOSITION 15. *The set $U_\alpha \cap \bar{Z}$ is open and dense in $\bar{Z}$.*

The proof of the proposition depends on the following lemma.

LEMMA 16. *Let $\gamma \in Z$ be given. Let α be a simple closed geodesic which does not cross γ. Then there is an orientation-preserving simple closed geodesic β which crosses α, and which crosses γ in a single point.*

Proof. Since γ is in Z, there is an orientation-preserving simple closed geodesic δ which meets γ in a single point. If δ crosses α then we are done. Otherwise, we write $\Sigma = A \cup T_0$, where A is a compact surface with one boundary component and T_0 is a 2-torus with a hole; we can do this so that $\alpha \subset A$ and $\gamma \cup \delta \subset T_0$. Since Σ is a non-exceptional surface, we have $\chi(A) < 0$.

Consider first the case where α is isotopic to ∂A. Since $\chi(A) < 0$ and A has only one boundary component, there is an orientation-preserving simple closed curve ω in A which is neither trivial nor boundary-parallel. In this case we take β to be the simple closed geodesic freely homotopic to the connected sum of δ with ω.

Now suppose that α is not boundary-parallel in A. Let $\hat{A}$ be the closed surface obtained by attaching a disk to ∂A. Then $\chi(\hat{A}) \leq 0$ and α is a homotopically non-trivial simple closed curve in $\hat{A}$. This implies that there is an orientation-preserving simple closed curve ω in $\hat{A}$ which has essential intersection with α. We again take β to be the simple closed geodesic freely homotopic to the connected sum of ω and δ.

In both cases, it is easy to see that β has all the properties asserted in Lemma 16. □

Proof of Proposition 15. First we show that U_α is an open subset of $\mathscr{P}L(\Sigma)$. Let V denote the set of all geodesics in $\mathbf{H}$ whose images in Σ cross α. Clearly, V is an open subset of $\mathscr{G}$. Hence, the set of measured geodesic laminations whose supports meet V is an open subset of $\mathscr{M}L(\Sigma)$. But the latter set is the pre-image of U_α in $\mathscr{M}L(\Sigma) - \{0\}$.

It follows immediately that $U_\alpha \cap \bar{Z}$ is an open subset of $\bar{Z}$. To show that it is dense, it is enough to show that every element of Z is in the closure of $Z \cap U_\alpha$.

Let $\gamma \in Z$ be given. If γ crosses α then $\gamma \in U_\alpha$ and we are done. Thus we may assume that α does not cross γ. Let β be the geodesic given by Lemma 16. Let β_n denote the simple closed geodesic freely homotopic to $\gamma^n * \beta$. By Proposition 14 the sequence (β_n) converges to γ in $\mathscr{P}L(\Sigma)$. We shall show that all but at most one of the β_n are contained in $U_\alpha \cap Z$, so that γ belongs to the closure of $U_\alpha \cap Z$ as required. We do this by proving that all the β_n are in Z and that all but at most one of them are in U_α.

Since β and γ are orientation-preserving and meet in a single point, each β_n meets γ in a single point. Thus, no β_n is trivial and no β_n represents w_1^*. Hence all the β_n are contained in Z.

If $\alpha = \gamma$, then all the β_n have intersection number ± 1 with α, and hence lie in U_α. Now suppose that $\alpha \neq \gamma$. This means that α and γ are disjoint. Let Σ_α denote the hyperbolic surface with totally geodesic boundary obtained by splitting Σ along α. We can write β as a union of a finite number of properly embedded geodesic arcs $b_1, \ldots, b_k$ in Σ_α. We take b_1 to be the arc containing the point of intersection of β and γ. Let $b_1 = c * d$ where c and d are arcs and $c \cap d = \beta \cap \gamma$. Then β_n is isotopic to a simple closed curve β'_n made up of the arcs $b_2, \ldots, b_k$ and an arc $b_1(n)$ which is homotopic in Σ_α, relative to its endpoints, to $c * \gamma^n * d$. If β_n does not cross α, then one of the arcs that make up β'_n must be homotopic in Σ_α, relative to its endpoints, into α. The only possible candidate for such an arc is $b_1(n)$. Suppose that there are distinct integers m and n such that β_m and β_n are not in U_α. By the above discussion, this means that $b_1(m)$ and $b_1(n)$ are both homotopic, relative to their endpoints, into $\partial\Sigma_\alpha$. It follows that γ^{n-m} is freely homotopic in Σ_α into the boundary. This is impossible since γ is a closed geodesic in int Σ_α. □

PROPOSITION 17. *On every closed non-exceptional hyperbolic surface Σ there exists a measured geodesic lamination whose leaves and complementary regions are all simply connected.*

Proof. Since $\bar{Z}$ is a non-empty compact Hausdorff space, it follows from Proposition 15 and the Baire Category Theorem [8, Thm. 2–78] that

$$\bar{Z} \cap \bigcap_{\alpha} U_\alpha \neq \varnothing$$

where α ranges over the set of simple closed geodesics on Σ. This means that there is a measured lamination $(\mathscr{L}, \mu)$ on Σ such that for every simple closed geodesic α there is a leaf of $\mathscr{L}$ crossing α. Thus, $\mathscr{L}$ contains no closed leaf, and there is no simple closed geodesic disjoint from the support of $\mathscr{L}$. Proposition 17 follows. □

Proof of Theorem 1. Let $(\mathscr{L}, \mu)$ be the measured lamination given by Proposition 17. By Lemma 8 there is a dual tree to $(\mathscr{L}, \mu)$ on which $\pi_1(\Sigma)$ acts. Since the leaves and complementary regions of $\mathscr{L}$ are simply connected it follows form Proposition 7 that the action is free. □

4. PROOF OF THEOREM 2

The purpose of this section is to prove Theorem 2. The result is clear for $\mathbf{R}P^2$ since any action of $\mathbf{Z}/2\mathbf{Z}$ on an $\mathbf{R}$-tree must have a fixed point. Suppose that G is isomorphic to the fundamental group of the Klein bottle and that we have a free action of G on an $\mathbf{R}$-tree. Since G contains a normal subgroup N isomorphic to $\mathbf{Z} \times \mathbf{Z}$, the minimal invariant sub-tree [5] for N is the minimal invariant sub-tree for G. But if T is an $\mathbf{R}$-tree admitting a minimal free action of $\mathbf{Z} \times \mathbf{Z}$, then T is isometric to $\mathbf{R}$. Thus, the supposed free action of G on an $\mathbf{R}$-tree contains a free action of G on $\mathbf{R}$. But it is easy to see that there is no free action of G on $\mathbf{R}$ by isometries.

The rest of this section treats the case when G is isomorphic to the fundamental group of a closed surface of Euler characteristic -1.

LEMMA 18. *Let F be a free group on generators a and b. Let*

$$F \times T_0 \to T_0$$

be a minimal free action on an $\mathbf{R}$-tree. Let $A \subset T_0$ be the axis of the commutator $[a, b]$. Let τ be the translation length of $[a, b]$. Then there exist a point $p \in A$ and an element $g \in F$ such that $gp \in A$ and $d(p, gp) = \tau/2$.

Proof. By [11] or Theorem 5.2 of [7], we know that this action is properly discontinuous. There are three possibilities for the quotient space Q. They are pictured in Fig. 2.

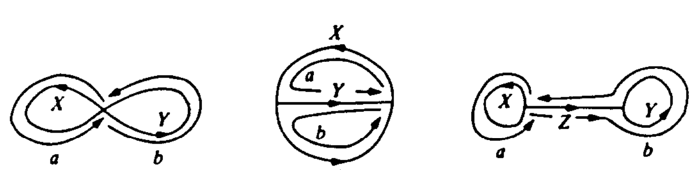

Fig. 2

Changing the basis for F leaves the commutator invariant up to conjugation and inversion. Thus, we can assume in each case that the basis $\{a, b\}$ for F is the geometric basis indicated in Fig. 2. (Here, we have chosen a base point in T and identified F with the fundamental group of Q.)

Since $[a, b]$ has an axis, there is an isometric immersion f of the circle $S = \mathbf{R}/\tau\mathbf{Z}$ into Q representing the conjugacy class of $[a, b]$. Since the axis of A is unique, the immersion f is unique up to rotations of S. To construct f it suffices to find a cyclically reduced edge path in Q representing the conjugacy class of $[a, b]$. To prove the lemma it suffices to find two points p_0 and p_1 in S separated by a distance $\tau/2$ and having the same image in Q. We define the edge paths and points in the three cases in the manner shown in Fig. 3. In Case 2 the points p_0 and p_1 are the midpoints of the edges labelled X and X^{-1}. □

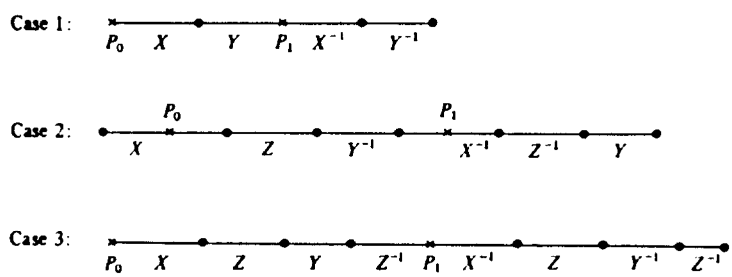

Fig. 3

Proof of Theorem 2 in the case of Euler characteristic -1. Let G be the fundamental group of a closed surface of Euler characteristic -1. We have a decomposition of G as a free product with amalgamation

$$F *_{\langle t \rangle} \langle \lambda \rangle$$

where F is the free group generated by a and b, and where t is identified with the commutator $[a, b]$ in F and with λ^2 in $\langle \lambda \rangle$.

Suppose that there is a free action of G on an $\mathbf{R}$-tree T. Let T_0 be the minimal invariant subtree for $F \subset G$. By Lemma 18 there exist a point p in the axis A of the commutator $[a, b]$ and an element $g \in F$ such that $gp \in A$ and $d(p, gp) = \tau/2$, where τ is the translation length of $[a, b]$.

On the other hand, in G we have $[a, b] = \lambda^2$. Thus, λ has A for its axis and has translation length $\tau/2$. Hence $\lambda^{\pm 1} gp = p$. This is a contradiction since $\lambda^{\pm 1} g$ is a non-trivial element of G. □

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APPENDIX

The purpose of this appendix is to give a proof of Proposition 14. Before directly broaching this result we need some elementary facts from hyperbolic geometry.

Lemma A1. *Let*

$$M = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

be an element of $SL(2, \mathbf{R})$ with trace > 2. Let A be its axis in $\mathbf{H}$. Then A crosses the positive imaginary axis I if and only if $qr > 0$. Furthermore, if $qr > 0$ then

$$\sqrt{qr} = \sin(\theta)\sinh(\tau/2)$$

where τ is the translation length of M and where θ is the acute angle between A and I.

Proof. The fixed points of M on the real axis are

$$\frac{(p-s) \pm \sqrt{(p-s)^2 + 4qr}}{2r}.$$

Thus A is the semi-circle in $\mathbf{H}$ with center $(p-s)/2r$ and radius $(\sqrt{(p-s)^2+4qr})/2r$. The first assertion follows. It also follows that if $qr > 0$, then

$$\sin(\theta) = \frac{2\sqrt{qr}}{\sqrt{(p-s)^2 + 4qr}}$$

which is equivalent to the second assertion. □

Lemma A2. *Let b and c be hyperbolic motions of $\mathbf{H}$ whose axes B and C cross at an angle θ and let τ be the translation length of b. Let $\varepsilon > 0$ be given. For every integer n set $b_n = c^n b$, let B_n and τ_n denote the axis and the translation length of b_n. Let S_n be the intersection of B_n with the ε-neighborhood of C. Let h_n be the hyperbolic length of the segment S_n. Then we have*

$$\lim_{n \to \infty} \tau_n = \infty \text{ and}$$

$$\lim_{n \to \infty} (\tau_n - h_n) = 2(\log\sinh(\tau/2) + \log\sin(\theta) - \log\tanh(\varepsilon)).$$

Proof. We can assume that a matrix representative for c is

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

with $\lambda > 1$, and consequently that C is the positive imaginary axis. Let

$$M = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{SL}(2, \mathbf{R})$$

be a matrix representative for b. By Lemma A1 we have $qr > 0$, and hence $p \neq 0$.

A matrix representative for B_n is

$$M_n = \begin{pmatrix} p_n & q_n \\ r_n & s_n \end{pmatrix} = \begin{pmatrix} \lambda^n p & \lambda^n q \\ \lambda^{-n} r & \lambda^{-n} s \end{pmatrix}.$$

Since $p \neq 0$ and $\lambda > 1$, we have $\lim_{n \to \infty} |\mathrm{trace}(M_n)| = \infty$. This implies the first assertion.

Since $q_n r_n = qr > 0$, it follows from Lemma A1 that for every n, the axes B_n and C cross, and that the angle θ_n formed by B_n and C satisfies

(A3) $$\sin(\theta_n)\sinh(\tau_n/2) = \sqrt{qr} = \sin(\theta)\sinh(\tau/2).$$

In light of this and the first assertion, we have

(A4) $$\lim_{n \to \infty} \theta_n = 0.$$

The point $B_n \cap C$ divides the segment S_n into two equal subsegments. Let T_n denote one of these. Then T_n is the hypotenuse of a hyperbolic right triangle having one leg in C; the other leg has length ε. (See Fig. 4.)

By formula 12.99 of [4] we have

(A5) $$\tanh(\varepsilon) = \sinh(h_n/2) \cdot \tan(\theta_n).$$

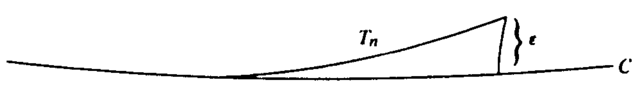

Fig. 4

It follows immediately from (A4) and (A5) that

(A6) $$\lim_{n \to \infty} h_n = \infty.$$

It follows from (A3) and (A5) that

(A7) $$\frac{\sinh(\tau_n/2)}{\sinh(h_n/2)} = \frac{\sin(\theta)\sinh(\tau/2)}{\tanh(\varepsilon) \cdot \cos(\theta_n)}.$$

The second conclusion of the lemma follows from (A4), (A6), and (A7). □

Proof of Proposition 14. The proposition asserts that

$$\lim_{n \to \infty} P(\beta_n) = P(\gamma).$$

Since $\mathscr{P}L(\Sigma)$ is compact, it is enough to show that any convergent subsequence of $(P(\beta_n))$ has $P(\gamma)$ as its limit. Let $(P(\beta_{n_i}))$ be a subsequence converging to some $P \in \mathscr{P}L(\Sigma)$. In order to show that $P = P(\gamma)$ it is enough to show that the support of P in $\mathscr{G}$ is contained in $L(\gamma)$, the set of geodesics covering γ.

Let $D \subset \mathbf{H}$ be a closed ball such that the translates of int D under $\pi_1(\Sigma, x)$ cover $\mathbf{H}$. Let $F \subset \mathscr{G}$ denote the set of all geodesics which meet D. Then F is compact and int F consists of all geodesics which meet int D. In particular the $\pi_1(\Sigma)$-translates of int F cover $\mathscr{G}$. Now define the set E and the map $\rho: \mathscr{M}L(\Sigma) \cap E \to \mathscr{P}L(\Sigma)$ as in the proof of Proposition 11. By Remark 13, ρ is a homeomorphism. Using the notation of Remark 13 we set $Q_i = Q(\beta_{n_i})$, $F_i = F(\beta_{n_i})$, and $N_i = \# F_i$. Note that $\lim_{i\to\infty} Q_i = Q$ where $Q = \rho^{-1}(P)|F$. Since $\operatorname{supp}(P)$ is $\pi_1(\Sigma)$-invariant, since $\operatorname{supp}(Q) = \operatorname{supp}(P) \cap F$ and since the $\pi_1(\Sigma)$-translates of int F cover $\mathscr{G}$, it suffices to show that $\operatorname{supp}(Q) \cap \operatorname{int} F$ is contained in $L(\gamma)$.

To do this we fix an arbitrary constant $\varepsilon > 0$; we shall show

(A8) *For every $l \in \operatorname{supp}(Q)$, the intersection $l \cap D$ is contained in the ε-neighborhood of a geodesic in $L(\gamma)$.*

This implies Proposition 14. For let $l \in \operatorname{supp}(Q) \cap \operatorname{int} F$. Since the union of the geodesics in $L(\gamma)$ is a closed subset of H, it follows from (A8) that $l \cap D$ is contained in a geodesic in $L(\gamma)$. Since $l \in \operatorname{int} F$, the intersection $l \cap D$ is a geodesic segment of positive length. Since this intersection is contained in a geodesic of $L(\gamma)$, it follows that $l \in L(\gamma)$.

It remains to establish (A8).

Let $x = \beta \cap \gamma$. Let b and c be the covering transformations determined by the loops β and γ based at x. Their axes B and C cross. For each integer n set $b_n = c^n b$, and let B_n and C denote the axes of b_n and c.

By Remark 13 we have

$$Q_i = \frac{1}{N_i} \sum_{l \in F_i} \delta_l.$$

By definition N_i is the number of translates of B_{n_i} that meet int D. Since by Lemma A2 the translation length of b_{n_i} tends to ∞ with i, it follows that

$$\lim_{i\to\infty} N_i = \infty. \tag{A9}$$

We denote by $\mathscr{C}_i$ the set of all $l \in F_i$ such that $l \cap D$ is contained in the ε-neighborhood of a geodesic in $L(\gamma)$. Let S_n be the intersection of B_n with the ε-neighborhood of C. It follows from Lemma A2 that there is a coarse fundamental domain J_n for the action of b_n on B_n such that $S_n \subset J_n$, and $K_n = J_n - S_n$ is an arc whose length is bounded as $n \to \infty$. Since the length of K_n is bounded as $n \to \infty$, the number of $\pi_1(\Sigma)$-translates of D that meet K_n is bounded. Thus the number of translates of K_n that meet D is bounded. But each geodesic in $F_i - \mathscr{C}_i$ contains a translate of K_{n_i} which meets D. Hence $\#(F_i - \mathscr{C}_i)$ is bounded as $i \to \infty$. Combining this with (A9) we conclude that

$$\lim_{i\to\infty} \frac{1}{N_i} \#(F_i - \mathscr{C}_i) = 0. \tag{A10}$$

Let us set

$$Q_i' = \frac{1}{N_i} \sum_{l \in \mathscr{C}_i} \delta_l.$$

Then by (A10) we have

$$\lim_{i\to\infty} Q_i' = \lim_{i\to\infty} Q_i = Q. \tag{A11}$$

By definition, for every $l \in \operatorname{supp}(Q_i')$, the intersection $l \cap D$ is contained in the ε-neighborhood of a geodesic in $L(\gamma)$. Thus, (A8) follows from (A11). □